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Discrete Optimization (ADM II)
SoSe 2012

Fourth Set of Exercises

(due date: May 17, **before** the exercise session)

Exercise 19 (3 points)**.** Given a graph $G = (V, E)$ and a node set $X \subseteq V$, we denote by $q_G(X)$ the number of odd connected components in the subgraph $G - X$ (obtained from G by deleting X and all edges incident to X). Show that $q_G(X) - |X| \equiv |V| \bmod 2$.

Exercise 20 (3+3 points)**.** Let M be a matching in G.

a) Show that there are at least $\nu(G) - |M|$ node-disjoint M-augmenting paths in G. (Recall that $\nu(G)$ denotes the size of a maximum matching in G).

b) Suppose that $\nu(G) = 5000$, and M is a matching of size 4000. Show that there exists at least one M-augmenting path with at most 9 edges.

Exercise 21 (2+3 points)**.** Recall that $G = (V, E)$ is a König-Egerváry graph (KEG, for short) iff $\nu(G) = \tau(G)$ (where $\tau(G)$ denotes the size of a minimum node cover in G).

a) Let M be a perfect matching in G. Show that, if G were a KEG, each minimum node cover would contain exactly one endpoint from each matching edge in M.

b) Now, let M be a maximum matching in G with $\nu(G) = |M| < \frac{1}{2}|V|$ (i.e., M is not perfect). Let $\hat{V} \subseteq V$ denote the set of M-exposed nodes. We expand G to a larger graph $G' = (V', E')$ with

$$V' = V \cup \{v' \mid v \in \hat{V}\}$$
$$E' = E \cup \{\{u, v'\} \mid v \in \hat{V},\ \{u, v\} \in E\} \cup \{\{v, v'\} \mid v \in \hat{V}\}$$

Note that $M' = M \cup \{\{v, v'\} \mid v \in \hat{V}\}$ is a perfect matching in G'. Show that G is a KEG if and only if G' is a KEG.

Exercise 22 (3+3 points)**.** KEGs have the nice property that a minimum node cover can be computed in polynomial time. By the previous exercise it suffices to consider graphs that admit a perfect matching. Let M be a perfect matching in G.

a) Sketch a polynomial-time algorithm that either returns a node cover of size $|M|$, or returns a certificate (an M-handcuff, as described below) proving that G is not a KEG.

b) An *M-handcuff* in G consists of two (not necessarily node-disjoint) M-blossoms whose bases are linked by an odd M-alternating path starting and ending with a matching edge. Show that G is a KEG if and only if G contains no M-handcuff.