Technische Universität Berlin
Institut für Mathematik
Dr. Britta Peis
Prof. Dr. Martin Skutella
Ágnes Cseh

Discrete Optimization (ADM II)
SoSe 2012

Third Set of Exercises

(due date: May 3, **before** the exercise session)

Exercise 15 (exercise session)**.** Consider an undirected graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}$. For $S \subseteq V$ let $\gamma(S) := \big\{e = \{v, w\} \in E \mid v, w \in S\big\}$. For a vector $x \in \mathbb{R}^E$ and a subset $B \subseteq E$ let $x(B) := \sum_{e\in B} x_e$. Last semester we proved that the following is an LP formulation of the minimum spanning tree problem (Theorem 5.6 and Corollary 5.7):

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & x(\gamma(S)) \leq |S| - 1 && \text{for all } \emptyset \neq S \subset V, \\ & x(E) = |V| - 1 \\ & x_e \geq 0 && \text{for all } e \in E. \end{aligned}$$

The goal of this exercise is to show that the separation problem of this linear program can be solved efficiently by setting up a maximum s-t-flow problem in a new digraph $D = (V', A)$. Let $V' := \{s, t\} \cup V \cup E$ and

$$A := \{(s, e) \mid e \in E\} \cup \{(e, v) \mid v \in V,\, e \in \delta(v)\} \cup \{(v, t) \mid v \in V\} \ .$$

For a given vector $x \in \mathbb{R}^E_{\geq 0}$ with $x(E) = |V| - 1$, define arc capacities on A by

$$\begin{aligned} u\big((s, e)\big) &:= x_e && \text{for } e \in E, \\ u\big((e, v)\big) &:= \infty && \text{for } v \in V,\, e \in \delta(v), \\ u\big((v, t)\big) &:= 1 && \text{for } v \in V. \end{aligned}$$

Finally, for node $v \in V$, let D_v be the digraph obtained by adding an infinite capacity arc (s, v) to D.

Prove that x fulfills all LP constraints if and only if, for all $v \in V$, the maximum s-t-flow value in D_v is equal to $|V|$.

Exercise 16 (2+3+2)**.** Consider the following variation of the diet problem: There are n foods and m nutrients where one unit of food j contains $a_{ij} \geq 0$ units of nutrient i. Consider a parent with two children with minimal nutritional requirements $b^1 \in \mathbb{R}^m_{\geq 0}$ and $b^2 \in \mathbb{R}^m_{\geq 0}$, respectively. Finally, let $c_j > 0$ be the cost of one unit of food j.

The parent has to buy food to satisfy the children's needs, at minimum cost. To avoid jealousy, there is the additional constraint that the amount to be spent for each child is the same.

a) Provide a standard form formulation of this problem. What are the dimensions of the constraint matrix?

b) If the Dantzig-Wolfe method is used to solve the problem in part a), construct the subproblems solved during a typical iteration of the master problem.

c) Suggest a direct approach for solving this problem based on the solution of two single-child diet problems.

Exercise 17 (3+3 points)**.** Consider a digraph $D = (V, A)$ with a source node s and a sink node t.

a) Consider the set of all s-t-flows $P := \{x \in \mathbb{R}^A \mid x \text{ is an } s\text{-}t\text{-flow}\}$. Give a complete characterization of the extreme points and the extreme rays of P (with proofs!).

b) For a given flow value $d > 0$ consider the set of all s-t-flows of value d, i. e., $Q := \{x \in \mathbb{R}^A \mid x \text{ is an } s\text{-}t\text{-flow of value } d\}$. Give a complete characterization of the extreme points and the extreme rays of Q (with proofs!).

Hint: Use the flow decomposition theorem (Theorem 6.14).

Exercise 18 (2+3+2 points)**.** The *Maximum Multi-Commodity Flow Problem* is defined as follows:

Given: Digraph $D = (V, A)$ with capacities $u : A \to \mathbb{R}_{\geq 0}$ and k commodities given by source-sink pairs $(s_i, t_i) \in V \times V$, for $i = 1, \dots, k$.

Task: Find s_i-t_i-flows x_i, for $i = 1, \dots, k$, obeying arc capacities $\sum_{i=1}^{k} x_{i,a} \leq u(a)$, for every arc $a \in A$, such that the sum of the flow values $\sum_{i=1}^{k} \mathrm{ex}_{x_i}(t_i)$ is maximal.

a) Formulate the Maximum Multi-Commodity Flow Problem as a linear program in arc variables.

b) Apply Dantzig-Wolfe decomposition where the arc capacity constraints are the only "coupling constraints". Determine the extreme points and extreme rays of the polyhedra corresponding to the subproblems and formulate the master problem.

c) Discuss how the pricing problem can be solved efficiently.