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Discrete Optimization (ADM II)
SoSe 2012

Second Set of Exercises

(due date: April 27, 4 pm, room MA 501)

Exercise 9 (6 points)**.** Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Consider the linear program in standard form

$$\begin{aligned} \min\quad & c^T \cdot x \\ \text{s.t.}\quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

and assume that the columns $A_1, \dots, A_m$ of A form an optimal basis. Now the first column A_1 is changed to $A_1 + \delta \cdot A_0$ for some $A_0 \in \mathbb{R}^m$ and $\delta \in \mathbb{R}$. Consider the matrix $B(\delta) \in \mathbb{R}^{m\times m}$ consisting of the columns $A_1 + \delta \cdot A_0, A_2, \dots, A_m$. Let $\delta_1 < 0 < \delta_2$ such that $B(\delta)$ is non-singular for all $\delta_1 \leq \delta \leq \delta_2$. Show that the subset of the interval $[\delta_1, \delta_2]$ for which $B(\delta)$ is an optimal basis is again a closed interval.

Exercise 10 (2+2+2 points)**.** For fixed $\theta \in \mathbb{R}_{\geq 0}$ consider the linear programming problem

$$\begin{aligned} \min\quad & x_1 + x_2 \\ \text{s.t.}\quad & x_1 + 2x_2 = \theta \\ & x_1, x_2 \geq 0 \end{aligned}$$

a) Find an optimal solution for all values of θ.

b) Draw a graph showing the optimal cost as a function of θ.

c) Use the picture in b) to obtain the set of all dual optimal solutions, for every value of θ.

Exercise 11 (3+2+3 points)**.** Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c, d \in \mathbb{R}^n \setminus \{0\}$. For $\theta \in \mathbb{R}$ consider the following parametric program:

$$\begin{aligned} g(\theta) := \min\quad & (c + \theta \cdot d)^T \cdot x \\ \text{s.t.}\quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

a) Let $\theta_1 < \theta_2$ and suppose that $g(\theta)$ is linear for $\theta \in [\theta_1, \theta_2]$. Is it true that there exists a unique optimal solution when $\theta_1 < \theta < \theta_2$? Prove or provide a counterexample.

b) Suppose that for some value of θ, there are exactly two distinct basic feasible solutions that are optimal. Is it true that they must be adjacent? Prove or provide a counterexample.

c) Let θ^* be a breakpoint of the function $g(\theta)$. Let x^1, x^2, x^3 be pairwise different basic feasible solutions, all of which are optimal for $\theta = \theta^*$. Suppose that x^1 is a unique optimal solution for $\theta < \theta^*$, x^3 is a unique optimal solution for $\theta > \theta^*$, and x^1, x^2, x^3 are the only basic feasible solutions for $\theta = \theta^*$. Provide an example to show that x^1 and x^3 need not be adjacent.

Exercise 12 (tutorial session)**.** For $\theta \in \mathbb{R}$, consider the following linear program:

$$\begin{array}{llllllll}
\min & 4\,x_1 & & + \; 5\,x_3 & & & \\
\text{s.t.} & 2\,x_1 & + \; x_2 & - \; 5\,x_3 & & = & 1 - 2\,\theta \\
 & -3\,x_1 & & + \; 4\,x_3 & + \; x_4 & = & 2 - 3\,\theta \\
 & & & x_1,\, x_2,\, x_3,\, x_4 & & \geq & 0
\end{array}$$

a) For fixed $\theta := 0$, write down a simplex tableau and find an optimal solution. Is it unique?

b) For fixed $\theta := 0$, write down the dual problem and find an optimal solution. Is it unique?

c) Find an optimal solution and the value of the optimal cost as a function of $\theta \in \mathbb{R}$.

Exercise 13 (tutorial session)**.** For $\delta \in \mathbb{R}$ consider the following linear program:

$$\begin{array}{llllllll}
\min & -5\,x_1 & - \; x_2 & + \; 12\,x_3 & & & \\
\text{s.t.} & (3+\delta)\,x_1 & + \; 2\,x_2 & + \; x_3 & & = & 10 \\
 & 5\,x_1 & + \; 3\,x_2 & & + \; x_4 & = & 16 \\
 & & & x_1,\, x_2,\, x_3,\, x_4 & & \geq & 0
\end{array}$$

For $\delta := 0$ set up a simplex tableau and solve the linear program (observe that x_1 and x_2 are the basic variables in an optimal solution). Let us keep x_1 and x_2 as the basic variables and let $B(\delta)$ be the corresponding basis matrix, as a function of δ (for small values of δ).

a) Compute $B(\delta)^{-1} \cdot b$. For which values of δ is $B(\delta)$ a feasible basis?

b) Compute ${c_B}^T \cdot B(\delta)^{-1}$. For which values of δ is $B(\delta)$ an optimal basis?

c) Determine the optimal cost, as a function of δ, when δ is restricted to those values for which $B(\delta)$ is an optimal basis matrix.

Exercise 14 (tutorial session)**.** Let $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Consider the following linear program in standard form:

$$\begin{array}{ll}
\min & c^T \cdot x \\
\text{s.t.} & A \cdot x = b \\
 & x \geq 0
\end{array}$$

Suppose that the first m columns $A_1, \ldots, A_m$ of A form an optimal basis $B \in \mathbb{R}^{m \times m}$ that leads to a non-degenerate optimal solution x^*, and a non-degenerate dual optimal solution p. We now change the first entry a_{11} of A_1 to $a_{11} + \delta$, for some $\delta \in \mathbb{R}$. Let $E \in \mathbb{R}^{m \times m}$ with entries all zero except for the top left entry e_{11} which is equal to 1.

a) Show that if δ is small enough, $B + \delta \cdot E$ is a basis matrix for the new problem.

b) Show that under the basis $B + \delta \cdot E$, the vector x_B of basic variables in the new problem is equal to $(I + \delta \cdot B^{-1} \cdot E)^{-1} \cdot B^{-1} \cdot b$.

c) Show that if δ is sufficiently small, $B + \delta \cdot E$ is an optimal basis for the new problem.