Technische Universität Berlin
Institut für Mathematik
Dr. Britta Peis
Prof. Dr. Martin Skutella
Ágnes Cseh

Discrete Optimization (ADM II)
SoSe 2012

First Set of Exercises

(due date: April 19, **before** the exercise session)

Exercise 1 (5 points)**.** Use Fourier-Motzkin-Elimination to determine whether or not the following system of linear inequalities has a solution (successively eliminate the variables x_1, x_2, and x_3):

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 - x_4 &\leq 1 \\ x_1 - 2x_2 + 3x_3 + x_4 &\leq 0 \\ -3x_1 + 6x_2 - 2x_3 - 3x_4 &\leq 2 \\ -x_1 - 3x_2 - 3x_3 - 3x_4 &\leq -3 \\ 3x_2 + x_3 + x_4 &\leq 2 \end{aligned}$$

Exercise 2 (3+2 points)**.** The aim of this exercise is to derive Farkas' Lemma from the Separating Hyperplane Theorem 9.3.
Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and consider the system $A \cdot x = b$, $x \geq 0$.

a) Let $A_1, \ldots, A_n \in \mathbb{R}^m$ be the columns of A and $S := \{\sum_{i=1}^n A_i \cdot x_i \mid x_i \geq 0, i = 1, \ldots, m\}$. Let y^k, $k = 1, 2, \ldots$, be an infinite sequence of elements of S that converges to some $y \in \mathbb{R}^m$. Prove that $y \in S$ (i. e., S is closed). Also argue that S is non-empty and convex.

b) Assume that the system $A \cdot x = b$, $x \geq 0$, has no solution. Apply the Separating Hyperplane Theorem to S and b and show that there is a vector y satisfying $y^T \cdot A \geq 0$ and $y^T \cdot b < 0$.

Hint for part a): Consider the problem of minimizing $\|y - \sum_{i=1}^n A_i \cdot x_i\|_\infty$ subject to the constraints $x_1, \ldots, x_n \geq 0$. Explain why this minimization problem has an optimal solution and determine the optimal solution value.

Exercise 3 (6 points)**.** Consider two non-empty polyhedra $P := \{x \in \mathbb{R}^n \mid A \cdot x \leq b\}$ and $Q := \{x \in \mathbb{R}^n \mid D \cdot x \leq d$. Show that $P \cap Q = \emptyset$ if and only if there is a vector $c \in \mathbb{R}^n$ such that $c^T \cdot x < c^T \cdot y$ for all $x \in P$ and $y \in Q$.

Hint: Formulate the problem of finding a point in $P \cap Q$ as a linear program. Consider the dual of this linear program.

Exercise 4 (4 points)**.** Show that every element x of a polytope $P \subseteq \mathbb{R}^n$ can be expressed as a convex combination of at most $n + 1$ extreme points of P.

Hint: Consider an extreme point of the set of all possible representations of x.

Remark: This result is known as *Carathéodory's Theorem.*

Exercise 5 (tutorial session)**.** Let $C \subseteq \mathbb{R}^n$ be a convex cone. A subset $S \subseteq C$ is a *generating system* for C if

$$C = \left\{ \sum_{i=1}^{k} \lambda_i \cdot x_i \mid k \in \mathbb{Z}_{\geq 0},\, x_1, \dots, x_k \in S,\, \lambda_1, \dots, \lambda_k \geq 0 \right\} .$$

If S is a minimal (w.r.t. inclusion) generating system, then S is a *basis of cone* C.

a) Give two different bases for the cone $\mathbb{R}^2$ with different cardinalities.

b) Do there exist convex cones in $\mathbb{R}^n$ with infinite bases?

Exercise 6 (tutorial session)**.** What are the extreme points and extreme rays of the following polyhedra:

$$P_1 := \{(x_1, x_2) \mid x_1 - x_2 = 0,\, x_1 + x_2 = 0\}$$
$$P_2 := \{(x_1, x_2) \mid 4x_1 + 2x_2 \geq 8,\, 2x_1 + x_2 \leq 8\}$$

Is it possible to express each element of P_2 as a convex combination of its extreme points plus a non-negative linear combination of its extreme rays? Is this compatible with the Resolution Theorem?

Exercise 7 (tutorial session)**.** Let P be a polyhedron with at least one extreme point. Is it possible to express an arbitrary element of P as a convex combination of its extreme points plus a non-negative multiple of a single extreme ray?

Exercise 8 (tutorial session)**.** Prove the following statements

a) $(\exists\, x : A \cdot x = c) \ \dot\vee\ (\exists\, y : A^T \cdot y = 0,\, c^T \cdot y = 1)$

b) $(\exists\, x : A \cdot x \leq c,\, A \cdot x \neq c)$
$\dot\vee\ \big[\exists\, y : (A^T \cdot y = 0,\, c^T \cdot y = -1,\, y \geq 0) \vee (A^T \cdot y = 0,\, c^T \cdot y \leq 0,\, y > 0)\big]$

c) $(\exists\, x : A \cdot x > 0,\, C \cdot x \geq 0,\, D \cdot x = 0) \ \dot\vee\ (\exists\, u, v, w : u, v \geq 0,\, u \neq 0,\, A^T \cdot u + C^T \cdot v + D^T \cdot w = 0)$

d) $(\exists\, x : A \cdot x \leq 0,\, x \neq 0) \ \dot\vee\ (\forall c : \exists\, y : y^T \cdot A = c,\, y \geq 0)$