Node-Arc Incidence-Matrices

Let us start with an example of network matrices:

Theorem 14.22 (Poincaré, 1900).

Let A be a matrix with entries in $\{-1, 0, 1\}$, where each column has at most one $+1$ and at most one -1. Then A is TUM.

Proof: ... □

Corollary 14.23.

Let $D = (V, E)$ be a directed graph and let A be the node-arc incidence matrix of D with entries

$$a_{ve} := \begin{cases} +1 & \text{if } v \text{ is the head of } e, \\ -1 & \text{if } v \text{ is the tail of } e, \\ 0 & \text{otherwise.} \end{cases}$$

Then A is TUM.

Example: ...

Network Matrices

Definition 14.24.

Let $D = (V, E)$ be a digraph and $T = (V, E')$ be a spanning tree od D. Then $M \in \{-1, 0, 1\}^{E' \times E}$ with entries

$$M_{e',(u,v)} := \begin{cases} +1 & \text{if the } (u,v)\text{-path in } T \text{ uses } e' \text{ in forward direction,} \\ -1 & \text{if the } (u,v)\text{-path in } T \text{ uses } e' \text{ in backward direction,} \\ 0 & \text{otherwise.} \end{cases}$$

is called network matrix (w.r.t. D and T).

Example: ...

Theorem 14.25 (Tutte, 1965).

Network Matrices are TUM.

Proof: ... □

Examples of Network Matrices

Observation 14.26.

Node-arc incidence matrices are network matrices.

Proof: . . . □

Observation 14.27.

Let $G = (V, E)$ be a bipartite graph and $M \in \{0,1\}^{V \times E}$ its node-edge incidence matrix with entries $M_{v,e} = 1$ iff v is incident to e. Then M is a network matrix.

Proof: . . . □

Observation 14.28.

A $\{0,1\}$-matrix A is said to have the consecutive-ones-property if the 1's in each column occur consecutively. Such a matrix is a network matrix.

Proof: Exercise. □

Total Dual Integrality

Min-max results on combinatorial objects can often be achieved by applying the LP-duality

$$\max\{w^T x \mid Ax \leq b\} \quad = \min\{y^T b \mid y^T A = w^T,\ y \geq 0\}. \qquad (14.1)$$

Definition 14.29.

A rational linear system $Ax \leq b$ is totally dual integral (TDI) if the minimum in (14.1) can be achieved by an integral vector y for each integral w for which optima exist.

Theorem 14.30 (Hoffman'74).

Let $Ax \leq b$ be a TDI-system with integral b such that $P = \{x \mid Ax \leq b\}$ is a rational polytope. Then P is integral.

Proof: . . . □

TDI-Systems always exist!

Theorem 14.31 (Giles/Pulleyblank'79).

Let P be a rational polyhedron. Then there exists a TDI-system $Ax \leq b$ with integral A such that $P = \{x \mid Ax \leq b\}$. Furthermore, if P is integral, then b can be chosen to be integral.

Proof: ommited. (Rough idea: Given $P = \{x \mid \tilde{A}x \leq \tilde{b}\}$, add a (redundant) inequality for each integral point that can be written as a nonnegative combination of the rows in $\tilde{A}$.) □

Possible approach to prove the integrality of polyhedra:

1. Find an appropriate defining system $Ax \leq b$ with A and b integral.
2. Prove that $Ax \leq b$ is TDI.
3. Conclude with Theorem 14.30 that $\{x \mid Ax \leq b\}$ is integral.

The usefulness of this plan strongly depends on methods for proving TDI-ness of systems.

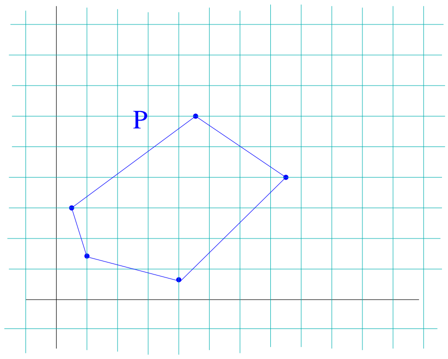
$\longrightarrow$ Important research branch!

Cutting Planes

Given an ILP of the form

$$(ILP) \quad \max\{w^T x \mid Ax \leq b,\ x \text{ integer}\},$$

where $P = \{x \mid Ax \leq b\}$ is *not integral*, we may still use linear-programming technique to find good solutions for this *special* w.



Observe: $x_2 \leq 5$ for all integral points in P.

Idea: Given an integral solution $\bar{x}$ of $Ax \leq b$ with $w^T\bar{x} = t$, in order to show the optimality of $\bar{x}$ we need to show that $w^T x \leq t$ holds for all integral vectors in $P = \{x \mid Ax \leq b\}$. Example: ...

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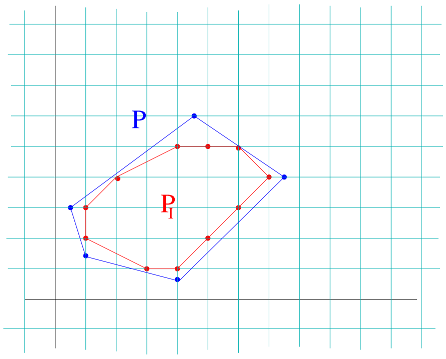

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Gomory-Chvátal Cutting Planes

Definition 14.32.

A cutting plane (or cut) of a polyhedron P is an inequality that is valid for all integral vectors in P.

Observation 14.33.

Let $P = \{x \mid a_i^T x \leq b_i (i = 1, \dots, m)\}$ and $y \in \mathbb{R}_+^m$. Then, for $c = \sum_{i=1}^m y_i a_i$ and $d = \sum_{i=1}^m y_i b_i$, the inequality $c^T x \leq \lfloor d \rfloor$ is a cutting plane.

If c is integral, then $c^T x \leq \lfloor d \rfloor$) is called Gomory-Chvátal cutting plane of P.

Gomory and Chvátal pioneered such a cutting plane approach to integer programming.