Sufficient Conditions for Integrality

Proving that polyhedra are integral is often a difficult task.

In the last decades, several conditions on system (A, b) have been developed that guarantee the integrality of $\{x \mid Ax \leq b\}$.

The probably most important concept here is *total unimodularity*.

To introduce the concept, we need the notion of *unimodular* matrices.

Lemma 14.15.

Let $A \in \mathbb{Z}^{m\times m}$ be an integral non-singular matrix. Then $A^{-1}b$ is integral for each integral vector $b \in \mathbb{R}^m \iff \det(A) \in \{-1, +1\}$.

Proof: . . . □

Unimodularity and Integrality

Definition 14.16.

A matrix A of full row rank is unimodular if A is integral and each basis of A has determinant -1 or 1.

Theorem 14.17 (Veinott/Dantzig 1968).

Let $A \in \mathbb{Z}^{m\times n}$ be a matrix of full row rank. Then $\{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}$ is integral for every integral vector $b \in \mathbb{R}^m \iff$ A is unimodular.

Proof: . . . □

Total Unimodularity

Definition 14.18.

Matrix A is totally unimodular (TUM) if all of its square submatrices have determinant $-1, 0$ or 1.

Thus, a TUM-matrix has all entries in $\{-1, 0, 1\}$.

Observation 14.19.

$A \in \mathbb{R}^{m \times n}$ is TUM $\iff$ $[A \; I_m]$ is unimodular.

Proof: Exercise. □

Theorem 14.20 (Hoffman/Kruskal 1956).

Let $A \in \mathbb{Z}^{m \times n}$. Then the polyhedron $\{x \in \mathbb{R}^n \mid Ax \leq b, \; x \geq 0\}$ is integral for every integral vector $b \in \mathbb{R}^m$ $\iff$ A is totally unimodular.

Proof: . . . □

Polyhedra without Non-Negativity Constraints

Note: For a given b it may be true that $\{x \mid Ax \leq b, \; x \geq 0\}$ is integral even if A is not TUM ($\longrightarrow$ Exercise.)

Theorem 14.21.

Let $A \in \mathbb{R}^{m \times n}$ be TUM and $b \in \mathbb{R}^m$ be integral. Then the polyhedron defined by $Ax \leq b$ is integral.

Proof: . . . □

This Theorem is often applied in order to prove the integrality of polyhedra. But how can we detect whether a given matrix is TUM?

In 1980, Seymour proved a deep Theorem which gives a polynomial-time test. In particulare, he showed that the backbone of every TUM-matrix are so-called network matrices, defined below.