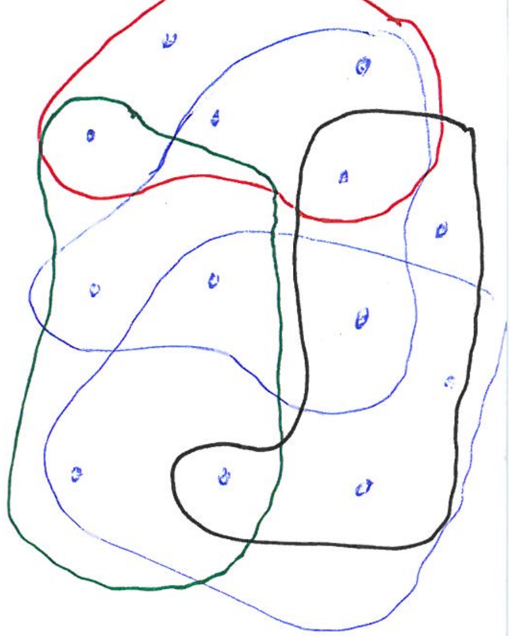
Set Cover Problem:



$$z^*_{IP} = \min \sum_{j=1}^{m} w_j x_j$$

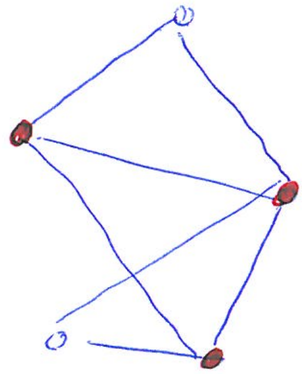
$$\sum_{j: e_i \in S_j} x_j \geq 1 \qquad i = 1, \dots, m$$

$$x_j \in \{0,1\} \qquad j = 1, \dots, m$$

Example Vertex Cover:

Given: Graph $G = (V, E)$, $w: V \to \mathbb{R}_+$

Task: Find a min-weight $W \subseteq V$ s.t. each edge $e \in E$ has at least one endpoint in W.



Lemma 17.3: The collection of subsets S_j with $j \in \hat{I} := \{j \mid x_j^* \geq \frac{1}{f}\}$ is a set cover.

Proof: We need to show that each $e_i \in E$ is "covered", i.e, that there exists some $j \in \hat{I} = \{j \mid x_j^* \geq \frac{1}{f}\}$ with $e_i \in S_j$.
Recall that $\sum_{j: e_i \in S_j} x_j^* \geq 1$ since x^* is feasible. Since there are $f_i \leq f$ terms in the sum, at least one term must be at least $\frac{1}{f}$. $\square$

Theorem 17.4: The rounding algorithm above is an f-approx. algo. for the set cover problem.

Proof: Clear that the algorithm runs in poly-time.
Note that $1 \leq f \cdot x_j^*$ for each $j \in \hat{I}$ by our construction. Thus,

$$w(\hat{I}) := \sum_{j \in \hat{I}} w_j \leq \sum_{j=1}^{m} w_j (f x_j^*) = f \sum_{j=1}^{m} w_j x_j^* = f \cdot z_{LP}^* \leq f \cdot \underbrace{z_{IP}^*}_{\text{value of optimal solution}} = f \cdot OPT \qquad \square$$

Dual rounding algo:

1. Compute an optimal solution y^* of dual problem
2. Select sets S_j with $j \in I^* := \{j \mid \sum y_i = w_j\}$

$$\begin{aligned} \max \; & \sum_{i=1}^{n} y_i \\ \text{s.t.} \; & \sum_{i: e_i \in S_j} y_i \leq w_j \quad \forall j = 1, \ldots, m \\ & y_i \geq 0 \quad \forall i = 1, \ldots, n \end{aligned}$$

Lemma 17.5: The collection of subsets S_j with $j \in I^*$ is a set cover.

Proof: Suppose e_k is not covered. Then for each S_j containing e_k we must have $\sum_{i: e_i \in S_j} y_i^* < w_j$. Let $\varepsilon := \min_{j: e_k \in S_j} (w_j - \sum_{i: e_i \in S_j} y_i^*) > 0$.

Consider vector y' with $y_i' = y_i^*$ for $i \neq k$ and $y_k' = y_k^* + \varepsilon$.

Then y' is a feasible dual solution by the definition of ε, and $\sum y_i' = \sum y_i^* + \varepsilon$, in contradiction to the optimality of y^*. $\square$

Theorem 17.6: The dual rounding algorithm is an f-approximation algorithm for the set cover problem.

Proof: "Charging arguements" (charge y_i^* for each i in S_j if S_j is selected)

$$w(I^*) = \sum_{j \in I^*} w_j = \sum_{j \in I^*} \sum_{i: e_i \in S_j} y_i^* = \sum_{i=1}^{m} |\{j \in I^* : e_i \in S_j\}| \cdot y_i^*$$

$$\leq \sum_{i=1}^{m} f_i y_i^* \leq f \cdot \sum_{i=1}^{m} y_i^* \leq f \cdot OPT. \quad \square$$