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Discrete Optimization (ADM II)
SoSe 2012

Ninth Set of Exercises

(due date: June 21, **before** the exercise session)

Exercise 53 (5 points)**.** Let $G = (V, E)$ be an undirected graph and consider the perfect matching polytope P defined by the inequalities

$$x(\delta(v)) = 1 \text{ for all } v \in V \tag{1}$$
$$x(\delta(U)) \geq 1 \text{ for all } U \subseteq V \text{ with } 3 \leq |U| \text{ odd.} \tag{2}$$
$$x_e \geq 0 \text{ for all } e \in E. \tag{3}$$

Given any subset $U \subseteq V$ with $3 \leq |U|$ odd, give a cutting-plane proof of $x(E(U)) \leq \frac{|U|-1}{2}$ starting from system (1)-(3).

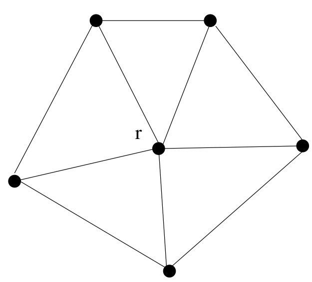
Exercise 54 (5 points)**.** Let $G = (V, E)$ be an undirected graph such that each node has degree at least one. Consider the polytope P defined by the inequalities

$$x_u + x_v \leq 1 \text{ for all } \{u, v\} \in E \tag{4}$$
$$x_v \geq 0 \text{ for all } v \in V. \tag{5}$$

Give a cutting-plane proof derived from this system for each of the following inequalities:

a) $x(C) \leq \frac{|C|-1}{2}$ for each set $C \subseteq V$ that forms an odd circuit in G.

b) $2x_r + x(W \setminus \{r\}) \leq 2$ for each set $W \subseteq V$ that forms a 5-wheel with root r.
(A 5-*wheel* is illustrated in Figure 54).



Exercise 55 (5 points)**.** Let $A \in \{0, 1\}^{m \times n}$ be a matrix whose row index set we denote by L, and whose column index set we denote by E. Assume that we are given a binary relation $\preceq$ on the row set L so that $(L, \preceq)$ forms a lattice $(L, \preceq, \wedge, \vee)$. Note that we can identify each row $i \in L$ with a subset $S_i \subseteq E$ in the sense that row i is the incidence vector of S_i. Given some function $r \in \mathbb{R}^m_+$ the polyhedron $P(A, r) = \{x \in \mathbb{R}^n_+ \mid Ax \geq r\}$ is called *lattice polyhedron* if the following holds for all $i, j, k \in L$:

a) $r(i) + r(j) \leq r(i \wedge j) + r(i \vee j)$,

b) $i \preceq j \preceq k$ implies $S_i \cap S_k \subseteq S_j$,

c) $S_{i \wedge j} \cup S_{i \vee j} \subseteq S_i \cup S_j$.

Show that the system $Ax \leq r,\ x \geq 0$ is TDI.

Hint: For each $i \in L$ let $z(i) := Z^{\gamma(i)}$ where Z is some constant with $Z > 2$ and $\gamma(i) := |\{j \in L : i \prec j\}|$. Show that the support of any optimal solution y^* of the dual linear program $\min_{y \geq 0}\{y^T r \mid y^T A \leq c^T\}$ that maximizes the function $z^T y$ over all optimal dual solutions forms a chain in $(L, \preceq)$.

Exercise 56 (5 points)**.** Give an example of a TSP instance on n nodes that does not satisfy the triangle inequality such that the cost of the solution produced by the Nearest Neighbor algorithm to the cost of an optimal tour can be arbitrarily large.

Exercise 57 (tutorial session)**.** Let P =conv.hull$(\{(0,0),(1,0),(\frac{1}{2},3)\}$. Find a system of linear inequalities that defines $P' = \{x \in P \mid x$ satiesfies every GC-cut for $P\}$.

Exercise 58 (tutorial session)**.** A *partially ordered set (poset)* is a tuple $P = (E, \leq)$ consisting of a finite set E together with some binary relation $\leq$ defined on E, so that the following three properties hold for all $x, y, z \in E$:

(i) $x \leq x$ ("reflexivity"),

(ii) $x \leq y$ and $y \leq x$ imply $x = y$ ("antisymmetry"),

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ ("transitivity").

Two elements $x, y \in E$ are called *comparable* if either $x \leq y$ or $y \leq x$ (or both). Otherwise, x and y are *incomparable.* A subset $C \subseteq E$ such that any two elements in C are comparable is called a *chain.* The *height* of a poset $(E, \leq)$ is the maximum cardinality of a chain in $(E, \leq)$.

A poset $(E, \leq)$ forms a *lattice* if for any two elements $x, y \in E$ there exists a unique *least common upper bound* $x \wedge y := \inf\{z \in E \mid x, y \leq z\}$, called *join* of x and y, as well as a unique *largest common lower bound* $x \vee y := \sup\{z \in E \mid x, y \leq z\}$, called *meet* of x and y.
Let S be a finite set.

a) Show that the family of all subsets of S ordered by inclusion forms a lattice (called *Boolean Lattice*). Determine the height of the lattice.

b) Show that $(S, |)$ such that $x|y$ iff x is a divisor of y forms a lattice.Determine the height of $(S, |)$.

Exercise 59 (tutorial session)**.**

Let $G = (V, E)$ be a connected graph with $s, t \in V$. Show that the family of all s,t-cuts $L(G) := \{\delta(S) \subseteq E \mid S \subseteq V \setminus \{t\},\ s \in S\}$ forms a lattice such that for all $C, C' \in L(G)$ holds

a) $(C \vee C') \cup (C \wedge C') \subseteq C \cup C'$,

b) $C \leq C' \leq C"$ implies $C \cup C" \subseteq C'$