Trivial Results using Complicated Methods

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27 September 2019
Berlin

joint work with
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Thanks to Elli for sharing her slides!
Tackling Vizing’s Conjecture — Road Map
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- A set $D \subseteq V$ is called a dominating set if for each $v \in V$
  - $v \in D$ or
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**Dominating Sets**

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The domination number $\gamma(G)$ is the cardinality of a minimum dominating set.
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- The domination number $\gamma(G)$ is the cardinality of a minimum dominating set.

![Diagram of a graph with nodes 1 to 9, illustrating no dominating set.](#)
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Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs.
Cartesian Product Graphs

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Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be graphs. The **Cartesian product graph** $G \Box H$ has

- vertex set $V(G) \times V(H)$ and
- the edge set

$$E(G \Box H) = \{(gh, g'h') : g = g' \text{ and } (h, h') \in E(H), \text{ or } h = h' \text{ and } (g, g') \in E(G)\}.$$
Cartesian Product Graphs

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Example 1

Example 2
Cartesian Product Graphs and Domination Number

Example 1

\[ \gamma(G) = 1 \]
\[ \gamma(H) = 1 \]
\[ \gamma(G \square H) = 2 \]

Example 2

\[ \gamma(G) = 2 \]
\[ \gamma(H) = 2 \]
\[ \gamma(G \square H) = 4 \]
Cartesian Product Graphs and Domination Number

Example 1

\[ \gamma(G) = 1 \]
\[ \gamma(H) = 1 \]
\[ \gamma(G \Box H) = 2 \]
\[ \gamma(G)\gamma(H) < \gamma(G \Box H) \]

Example 2

\[ \gamma(G) = 2 \]
\[ \gamma(H) = 2 \]
\[ \gamma(G \Box H) = 4 \]
\[ \gamma(G)\gamma(H) = \gamma(G \Box H) \]
Vizing's Conjecture

Conjecture (Vizing, 1968)

Given graphs $G$ and $H$, then the inequality

$$\gamma(G) \gamma(H) \leq \gamma(G \Box H)$$

holds.
Glimpse at History of Vizing’s Conjecture

- **1968**: Vizing
  - proposes conjecture

- **1979**: Barcalkin and German
  - ✓ for decomposable graphs

- **1990**: Faudree, Schelp and Shreve
  - ✓ for graphs that satisfy a special “coloring property”

- **2000**: Clark and Suen
  - $\gamma(G)\gamma(H) \leq 2\gamma(G\Box H)$

- **2003**: Sun
  - ✓ for graphs with $\gamma(G) \leq 3$

- **2009**: Bresar, Dorbec, Goddard, Hartnell, Henning, Klavzar, Rall
  - summarize properties of a minimum counterexample

- **2019**: Zerbib
  - $\gamma(G)\gamma(H) + \max\{\gamma(G),\gamma(H)\} \leq 2\gamma(G\Box H)$
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Let $G$ and $H$ be graphs. Then

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## Split Vizing’s Conjecture

### Conjecture (Vizing, 1968)

Let $G$ and $H$ be graphs. Then

\[ \gamma(G) \gamma(H) \leq \gamma(G \Box H). \]

### Definition

- For given $n_G, k_G \in \mathbb{N}$ with $k_G \leq n_G$
  - let $G$ be the class of graphs with
    - $n_G$ vertices and
    - domination number $k_G$. 

Split Vizing’s Conjecture

Conjecture (Vizing, 1968)

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- For given $n_H, k_H \in \mathbb{N}$ with $k_H \leq n_H$
  - let $H$ be the class of graphs with
    - $n_H$ vertices and
    - domination number $k_H$. 
Split Vizing’s Conjecture

**Conjecture (Vizing, 1968)**

Let $G$ and $H$ be graphs. Then

$$\gamma(G) \gamma(H) \leq \gamma(G \Box H).$$

**Observation**

Vizing’s conjecture holds iff

- for all values of $n_G, k_G, n_H, k_H \in \mathbb{N}$ with $k_G \leq n_G$ and $k_H \leq n_H$
  - for all $G \in \mathcal{G}$ and for all $H \in \mathcal{H}$
    - $\gamma(G) \gamma(H) \leq \gamma(G \Box H)$ holds.
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\[ \gamma(G) \gamma(H) \leq \gamma(G \boxplus H) \]

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## General Introduction: Ideals and Varieties

### Definition

- Let $P[x] = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables over the field $\mathbb{K}$.
- A subset $I \subseteq P[x]$ is called an **ideal** of $P[x]$ if it satisfies
  - $0 \in I$,
  - if $a, b \in I$, then $a + b \in I$,
  - if $a \in I$ and $b \in P[x]$, then $a \cdot b \in I$.

### Variety of an Ideal

- The **variety** of the ideal $I$ is defined as the set $V(I) = \{z \in \mathbb{K}^n : f(z) = 0 \text{ for all } f \in I\}$ with $\mathbb{K}$ being the algebraic closure of $\mathbb{K}$. 
<table>
<thead>
<tr>
<th>Definition</th>
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Definition

- Let \( I \subseteq P[x] \) be an ideal of the polynomial ring \( P[x] \).
- The variety of the ideal \( I \) is defined as the set

\[
\mathcal{V}(I) = \{ z \in \overline{\mathbb{K}}^n : f(z) = 0 \text{ for all } f \in I \}
\]

with \( \overline{\mathbb{K}} \) being the algebraic closure of \( \mathbb{K} \).
Ideal $I_G$

Definition

\[ \text{Let } n_G, k_G \in \mathbb{N} \text{ with } k_G \leq n_G. \]

\[ \text{Let } V(G) = \{1, 2, \ldots, n_G\} \text{ and } D_G = \{1, 2, \ldots, k_G\}. \]

\[ \text{Let } e_G = \{e_{gg'} : \{g, g'\} \subseteq V(G)\}. \]

The ideal $I_G \subseteq \mathbb{K}[e_G]$ is defined by the system of equations

\[ e_{gg'}^2 - e_{gg'} = 0 \quad \forall \{g, g'\} \subseteq V(G) \]

\[ \prod_{g' \in D_G} (1 - e_{gg'}) = 0 \quad \forall g \in V(G) \setminus D_G, \]

\[ \prod_{g' \in V(G)} (\sum_{g \in S} e_{gg'}) = 0 \quad \forall S \subseteq V(G) \text{ with } |S| = k_G - 1. \]

Theorem (Gaar, Krenn, Margulies, W. 2019)

The points in the variety $V(I_G)$ are in bijection to the graphs in $G$. 

\[ e_{gg'} = \begin{cases} 1 & \text{if } \{g, g'\} \text{ is an edge} \\ 0 & \text{otherwise} \end{cases} \]
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The points in the variety $\mathcal{V}(I_G)$ are in bijection to the graphs in $G$. 
Ideal $I_{G\Box H}$

Definition

- Let $n_G, k_G, n_H, k_H \in \mathbb{N}$ with $k_G \leq n_G$ and $k_H \leq n_H$. 

$$x_{gh} = \begin{cases} 
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Definition

- Let $n_G, k_G, n_H, k_H \in \mathbb{N}$ with $k_G \leq n_G$ and $k_H \leq n_H$.
- Let $x_{G \square H} = \{x_{gh} : g \in V(G), h \in V(H)\}$.
### Ideal $I_{G \square H}$

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Ideal \( I_{G \Box H} \)

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- Let \( n_G, k_G, n_H, k_H \in \mathbb{N} \) with \( k_G \leq n_G \) and \( k_H \leq n_H \).
- Let \( x_{G \Box H} = \{ x_{gh} : g \in V(G), h \in V(H) \} \).
- The ideal \( I_{G \Box H} \subseteq \mathbb{K}[e_G, e_H, x_{G \Box H}] \) is defined by

\[
\begin{align*}
    x_{gh}^2 - x_{gh} &= 0, \\
    (1 - x_{gh}) \left( \prod_{g' \in V(G), g' \neq g} (1 - e_{gg'} x_{g'h}) \right) \left( \prod_{h' \in V(H), h' \neq h} (1 - e_{hh'} x_{gh'}) \right) &= 0.
\end{align*}
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for all \( g \in V(G) \) and all \( h \in V(H) \).

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- Let $n_G, k_G, n_H, k_H \in \mathbb{N}$ with $k_G \leq n_G$ and $k_H \leq n_H$.
- Let $I_{\text{viz}}$ be the ideal generated by the generators of $I_G$, $I_H$ and $I_{G \Box H}$.
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**Theorem (Gaar, Krenn, Margulies, W. 2019)**

The points in the variety $\mathcal{V}(I_{\text{viz}})$ are in bijection to the triples $(G, H, D)$ where
Definition

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The points in the variety $\mathcal{V}(I_{viz})$ are in bijection to the triples $(G, H, D)$ where

- $G$ is a graph on $n_G$ vertices with $\gamma(G) = k_G$, 

where $\gamma(G)$ denotes the domination number of the graph $G$. 

$\square$
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- $H$ is a graph on $n_H$ vertices with $\gamma(H) = k_H$. 

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- Let $I_{\text{viz}}$ be the ideal generated by the generators of $I_G$, $I_H$ and $I_{G \Box H}$.

**Theorem (Gaar, Krenn, Margulies, W. 2019)**

The points in the variety $\mathcal{V}(I_{\text{viz}})$ are in bijection to the triples $(G, H, D)$ where

- $G$ is a graph on $n_G$ vertices with $\gamma(G) = k_G$,
- $H$ is a graph on $n_H$ vertices with $\gamma(H) = k_H$,
- $D$ is a dominating set of any size in $G \Box H$.  

Back to Vizing’s Conjecture

Definition

Given the graph classes $\mathcal{G}$ and $\mathcal{H}$, define

$$f^* = \left( \sum_{gh \in V(\mathcal{G}) \times V(\mathcal{H})} x_{gh} \right) - k_G k_H.$$
Definition

Given the graph classes $G$ and $H$, define

$$f^* = \left( \sum_{gh \in V(G) \times V(H)} x_{gh} \right) - k_G k_H.$$ 

Theorem (Gaar, Krenn, Margulies, W. 2019)

Vizing’s conjecture is true if and only if

- for all values of $n_G, k_G, n_H, k_H \in \mathbb{N}$ with $k_G \leq n_G$ and $k_H \leq n_H$

  $$f^*(z) \geq 0 \quad \forall z \in \mathcal{V}(l_{viz}).$$
Tackling Vizing’s Conjecture — Road Map

1 - Graph Theory
\[ \gamma(G) \gamma(H) \leq \gamma(G \Box H) \]

2 - Split
\[ n_G, k_G, n_H, k_H \]

3 - Algebraic Model

4 - Sum of Squares

5 - Semidefinite Programming

6 - SDP Solver

7 - Guess

8 - SAGE

9 - Prove

10 - Generalize
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General Introduction: Sum of Squares (SOS)

**Definition**

- Let \( I \) be an ideal of the polynomial ring \( P[x] \).
- Let \( \ell \) be a nonnegative integer.
- A polynomial \( f \in P[x] \) is called \( \ell \)-sum-of-squares modulo \( I \) (or \( \ell \)-sos mod \( I \)), if
  - there exist polynomials \( s_1, \ldots, s_t \in P[x] \) with degrees \( \deg s_i \leq \ell \) for all \( i \in \{1, \ldots, t\} \) and
    \[
    f \equiv \sum_{i=1}^{t} s_i^2 \mod I
    \]
    \[
    \iff f = \sum_{i=1}^{t} s_i^2 + g \text{ for some } g \in I.
    \]
In general:

- nonnegative on the variety $\iff$ being $\ell$-sos mod ideal
- nonnegative on the variety $\not\iff$ being $\ell$-sos mod ideal
Back to Vizing’s Conjecture

- In general:
  - nonnegative on the variety $\iff$ being $\ell$-sos mod ideal
  - nonnegative on the variety $\not\Rightarrow$ being $\ell$-sos mod ideal

- However, our specific setting:

---

**Theorem (Gaar, Krenn, Margulies, W. 2019)**

*Vizing’s conjecture is true if and only if*

- for all values of $n_G, k_G, n_H, k_H \in \mathbb{N}$ with $k_G \leq n_G$ and $k_H \leq n_H$
- $\exists \ell \in \mathbb{Z}$ such that $f^*$ is $\ell$-sos mod $l_{viz}$.

- If $f^* \equiv \sum_{i=1}^{t} s_i^2 \mod l_{viz}$, we call the $s_i$ a certificate
Tackling Vizing’s Conjecture — Road Map

1 - Graph Theory
\[ \gamma(G)\gamma(H) \leq \gamma(G \square H) \]

2 - Split
\[ n_G, k_G, n_H, k_H \]

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Semidefinite Programming

- Standard procedure to check if $f^*$ is $\ell$-sos mod $I_{\text{viz}}$:
  - fix a reduced Gröbner basis $B$ of $I_{\text{viz}}$
Semidefinite Programming

- Standard procedure to check if $f^*$ is $\ell$-sos mod $I_{\text{viz}}$:
  - fix a reduced Gröbner basis $B$ of $I_{\text{viz}}$
  - let $v$ be the vector of all monomials of degree at most $\ell$ which cannot be reduced over $B$

Observation (see e.g. Blekherman, Parrilo, Thomas 2012)

- Then $f^*$ is $\ell$-sos modulo $I_{\text{viz}}$ if and only if there is a positive semidefinite matrix $X \in \mathbb{R}^{p \times p}$ such that $f^*$ is equal to $v^T X v$ when reduced over $B$. 
Semidefinite Programming

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Semidefinite Programming

- Standard procedure to check if $f^*$ is $\ell$-sos mod $I_{\text{viz}}$:
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  - let $p$ be the length of the vector $\nu$

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    when reduced over $B$.

- Explanation:
  - $X \succeq 0 \Rightarrow \exists S : X = S^T S$
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- $X \succeq 0 \Rightarrow \exists S : X = S^T S$
- hence: $f^* \equiv v^T X v = (Sv)^T (Sv) = \sum_i s_i^2 \mod I_{\text{viz}}$
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Tackling Vizing’s Conjecture — Road Map

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Using an SDP Solver

- Semidefinite Program to check if $f^*$ is $\ell$-sos mod $I_{\text{viz}}$:
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If SDP infeasible:

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  $\Rightarrow$ increase $\ell$
Using an SDP Solver

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    - increase $\ell$
- If optimal solution found:
Using an SDP Solver

- Semidefinite Program to check if $f^*$ is $\ell$-sos mod $I_{\text{viz}}$:
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- Use an SDP Solver (e.g. MOSEK)
- If SDP infeasible:
  - $f^*$ is most likely not $\ell$-sos mod $I_{\text{viz}}$
  - $\leftarrow$ increase $\ell$
- If optimal solution found:
  - $f^*$ is most likely $\ell$-sos mod $I_{\text{viz}}$
  - in a perfect (exact) world:
    - numeric SDP solution for $X$ is exact
    - calculate eigenvalues decomposition $X = V^T D V$
    - set $S_i,j = D_i^1/2 V_i$
    - $S_i,j$ is the coefficient of the $j$-th monomial in the $i$-th polynomial $s_i$ of the exact sum-of-squares certificate
    - $f^* \sum s_i^2 \mod I_{\text{viz}}$
Using an SDP Solver

- Semidefinite Program to check if $f^*$ is $\ell$-sos mod $I_{\text{viz}}$:
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Using an SDP Solver

- Semidefinite Program to check if $f^*$ is $\ell$-sos mod $l_{\text{viz}}$:
  - matrix variable $X \in \mathbb{R}^{p \times p}$, $X \succeq 0$
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- Use an SDP Solver (e.g. MOSEK)
- If SDP infeasible:
  - \( f^* \) is most likely not \( \ell \)-sos mod \( I_{\text{viz}} \)
  - \( \rightarrow \) increase \( \ell \)
- If optimal solution found:
  - \( f^* \) is most likely \( \ell \)-sos mod \( I_{\text{viz}} \)
  - in a not perfect (exact) world:
    - numeric SDP solution for \( X \) is not exact
    - calculate eigenvalues decomposition \( X = V^T D V \) not exactly
    - set \( S = D^{1/2} V \)
    - \( S_{i,j} \) is the coefficient of the \( j \)-th monomial in the \( i \)-th polynomial \( s_i \) of the not exact sum-of-squares certificate
    - \( f^* \approx \sum_i s_i^2 \mod I_{\text{viz}} \)
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Example

\[ n_G = 3, \ k_G = 2, \ n_H = 3, \ k_H = 2 \]
Example

- $n_G = 3$, $k_G = 2$, $n_H = 3$, $k_H = 2$
- Construct the ideal $I_{viz}$
  - 15 variables ($3 e_G$, $3 e_H$, $9 x_{G \square H}$)
  - generated by 32 polynomials ($7 I_G$, $7 I_H$, $18 I_{G \square H}$)
  - reduced Gröbner basis of size 95
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  - reduced Gröbner basis of size 95
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  - $X \in \mathbb{R}^{12 \times 12}$
  - 67 linear equality constraints
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  - reduced Gröbner basis of size 95
- Is $f^*$ 1-sos mod $I_{\text{viz}}$?
  - $X \in \mathbb{R}^{12 \times 12}$
  - 67 linear equality constraints
  - infeasible
Example

- $n_G = 3$, $k_G = 2$, $n_H = 3$, $k_H = 2$
- Construct the ideal $I_{\text{viz}}$
  - 15 variables ($3 e_G$, $3 e_H$, $9 x_{G \Box H}$)
  - generated by 32 polynomials ($7 I_G$, $7 I_H$, $18 I_{G \Box H}$)
  - reduced Gröbner basis of size 95
- Is $f^*$ 1-sos mod $I_{\text{viz}}$?
  - $X \in \mathbb{R}^{12 \times 12}$
  - 67 linear equality constraints
  - infeasible
- Is $f^*$ 2-sos mod $I_{\text{viz}}$?
  - $X \in \mathbb{R}^{67 \times 67}$
  - 359 linear equality constraints
Example

- \( n_G = 3, \ k_G = 2, \ n_H = 3, \ k_H = 2 \)
- Construct the ideal \( I_{\text{viz}} \)
  - 15 variables (3 \( e_G \), 3 \( e_H \), 9 \( x_{G\boxtimes H} \))
  - generated by 32 polynomials (7 \( I_G \), 7 \( I_H \), 18 \( I_{G\boxtimes H} \))
  - reduced Gröbner basis of size 95
- Is \( f^* \) 1-sos mod \( I_{\text{viz}} \)?
  - \( X \in \mathbb{R}^{12 \times 12} \)
  - 67 linear equality constraints
  - infeasible
- Is \( f^* \) 2-sos mod \( I_{\text{viz}} \)?
  - \( X \in \mathbb{R}^{67 \times 67} \)
  - 359 linear equality constraints
  - SDP solution time: 0.72 seconds
  - optimal solution found 😊
  - start to guess the exact certificate!
Example: $n_G = 3, k_G = 2, n_H = 3, k_H = 2, X \in \mathbb{R}^{67 \times 67}$

- Plot of the entries of matrix $S = D^{1/2}V$
- $S_{i,j}$ is the coefficient of the $j$-th monomial in the $i$-th polynomial $s_i$ of the numeric sum-of-squares certificate
Example: \( n_G = 3, \ k_G = 2, \ n_H = 3, \ k_H = 2, \ X \in \mathbb{R}^{67 \times 67} \)

- Plot of the entries of matrix \( S = D^{1/2} V \)
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- Strategy:
  - play with objective function

This leads to a nice certificate:

4 (number of rows) polynomials 

\( s_1, \ldots, s_4 \)

in 19 (number of columns) monomials

nice block structure
Example: $n_G = 3$, $k_G = 2$, $n_H = 3$, $k_H = 2$

Strategy:

- play with objective function
- restrict to a subset of monomials
  
  use only the 19 monomials of the form
  
  $1$, $x_{gh}$ and $x_{gh}x_{gh'}$ for all $g \in V(G)$ and all $h, h' \neq h \in V(H)$

This leads to a nice certificate:

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Strategy:

- play with objective function
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This \( S \) leads to a nice certificate:

- 4 (number of rows) polynomials \( s_1, \ldots, s_4 \)
- in 19 (number of columns) monomials
- nice block structure
For $n_G = 3$, $k_G = 2$, $n_H = 3$ and $k_H = 2$

Vizing’s conjecture is true as the polynomials

\[ s_i = \nu_i + \sum_{g \in V(G)} \lambda_{g,i} \left( \sum_{h \in V(H)} x_{gh} \right) + \sum_{g \in V(G)} \mu_{g,i} \left( \sum_{\{h,h'\} \subseteq V(H)} x_{gh} x_{gh'} \right) \]

for $i \in \{1, \ldots, n_G\}$ and

\[ s_0 = \alpha + \beta \left( \sum_{g \in V(G)} \sum_{h \in V(H)} x_{gh} \right) + \gamma \left( \sum_{g \in V(G)} \sum_{\{h,h'\} \subseteq V(H)} x_{gh} x_{gh'} \right), \]

where $\alpha$, $\beta$, $\gamma$, $\nu_i$, $\lambda_{g,i}$ and $\mu_{g,i}$ are the entries of $S$, are a 2-sos certificate of $f^*$. 
Example: $n_G = 3$, $k_G = 2$, $n_H = 3$, $k_H = 2$

$$S = \begin{pmatrix} 0.535 & 0.011 & 0.011 & 0.011 & -0.289 & -0.289 & -0.289 \\ 0.000 & 0.000 & 0.236 & -0.236 & -0.001 & -0.471 & 0.472 \\ -0.000 & -0.272 & 0.136 & 0.136 & 0.544 & -0.273 & -0.272 \\ 2.778 & -0.962 & -0.962 & -0.962 & 0.536 & 0.536 & 0.536 \end{pmatrix}$$

- Entries of $S$ are hard to guess! 😞
Example:  $n_G = 3$, $k_G = 2$, $n_H = 3$, $k_H = 2$

$X = \begin{pmatrix}
-2.667 & 1.000 & 0.889 & 0.889 & -0.667 & -0.444 & -0.444 \\
-2.667 & 0.889 & 1.000 & 0.889 & -0.444 & -0.667 & -0.444 \\
-2.667 & 0.889 & 0.889 & 1.000 & -0.444 & -0.444 & -0.667 \\
1.333 & -0.667 & -0.444 & -0.445 & 0.667 & 0.222 & 0.222 \\
1.333 & -0.444 & -0.667 & -0.445 & 0.222 & 0.667 & 0.222 \\
1.333 & -0.444 & -0.444 & -0.667 & 0.222 & 0.222 & 0.667 \\
\end{pmatrix}$

- Entries of $X$ are easy to guess! 😊
- e.g. $0.667 = 2/3$
- Obtain guessed exact values for $X$
Example: $n_G = 3, k_G = 2, n_H = 3, k_H = 2$

- Have guessed exact values for $X$
- Use $S^T S = X$
Example: $n_G = 3, k_G = 2, n_H = 3, k_H = 2$

- Have guessed exact values for $X$
- Use $S^T S = X$
- Group coefficients of $S$
  \[
  \nu = (\nu_i)_{i=1, \ldots, n_G} \quad \mu_g = (\mu_{g,i})_{i=1, \ldots, n_G} \quad \lambda_g = (\lambda_{g,i})_{i=1, \ldots, n_G}
  \]
  \[
  a = \begin{pmatrix} \nu \\ \alpha \end{pmatrix} \quad b_g = \begin{pmatrix} \lambda_g \\ \beta \end{pmatrix} \quad c_g = \begin{pmatrix} \mu_g \\ \gamma \end{pmatrix}
  \]
- Obtain a system of equations
  \[
  \langle a, a \rangle = 2(n_G - 1)^2 \quad \langle b_g, b_g \rangle = 1 \quad \langle b_g, b_g' \rangle = \frac{8}{3}
  \]
  \[
  \langle a, b_g \rangle = -\frac{4}{3}(n_G - 1) \quad \langle c_g, c_g \rangle = \frac{6}{9} \quad \langle c_g, c_g' \rangle = \frac{2}{9}
  \]
  \[
  \langle a, c_g \rangle = \frac{2}{3}(n_G - 1) \quad \langle b_g, c_g \rangle = -\frac{6}{9} \quad \langle b_g, c_g' \rangle = -\frac{4}{9}
  \]
- If the guess for $X$ was correct:
  - each solution to the system of equations yields a certificate!
  - found an easy solution with $\nu = 0$
  - can be used to further simplify the certificate
Example: $n_G = 3$, $k_G = 2$, $n_H = 3$, $k_H = 2$

Conjecture

- For $n_G = 3$, $k_G = 2$, $n_H = 3$ and $k_H = 2$
  
  Vizing’s conjecture is true as the polynomials

\[
s_g = \frac{1}{3} \left( \sum_{h \in V(H)} x_{gh} - 2 \sum_{\{h, h'\} \subseteq V(H)} x_{gh} x_{gh'} \right) \quad \text{for } g \in V(G)
\]

\[
s_0 = \alpha + \beta \left( \sum_{g \in V(G)} \sum_{h \in V(H)} x_{gh} \right) + \gamma \left( \sum_{g \in V(G)} \sum_{\{h, h'\} \subseteq V(H)} x_{gh} x_{gh'} \right)
\]

where $\alpha = \sqrt{2}(n_G - 1)$, $\beta = -\frac{2}{3} \sqrt{2}$ and $\gamma = \frac{1}{3} \sqrt{2}$

are a 2-sos certificate of $f^*$. 
# Tackling Vizing’s Conjecture — Road Map

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1. **Graph Theory**
   \(\gamma(G)\gamma(H) \leq \gamma(G\Box H)\)

2. **Split**
   \(n_G, k_G, n_H, k_H\)

3. **Algebraic Model**
   \(f^*(z) \geq 0 \ \forall z \in \mathcal{V}(l_{viz})\)

4. **Sum of Squares**
   \(f^* \equiv \sum_{i=1}^{t} s_i^2 \mod l_{viz}\)

5. **Semidefinite Programming**
   \(\exists X : X \succeq 0, \ f^* \equiv v^T X v \mod l_{viz}\)

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Example: $n_G = 3$, $k_G = 2$, $n_H = 3$, $k_H = 2$

Theorem (Gaar, Krenn, Margulies, W. 2019)

For $n_G = 3$, $k_G = 2$, $n_H = 3$ and $k_H = 2$

Vizing’s conjecture is true as the polynomials

\[
    s_g = \frac{1}{3} \left( \sum_{h \in V(H)} x_{gh} - 2 \sum_{\{h, h'\} \subseteq V(H)} x_{gh} x_{gh'} \right) \text{ for } g \in V(G)
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    s_0 = \alpha + \beta \left( \sum_{g \in V(G)} \sum_{h \in V(H)} x_{gh} \right) + \gamma \left( \sum_{g \in V(G)} \sum_{\{h, h'\} \subseteq V(H)} x_{gh} x_{gh'} \right)
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where $\alpha = \sqrt{2}(n_G - 1)$, $\beta = -\frac{2}{3} \sqrt{2}$ and $\gamma = \frac{1}{3} \sqrt{2}$

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\[ \gamma(G) \gamma(H) \leq \gamma(G \square H) \]

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\[ n_g, k_g, n_H, k_H \]

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Example: \( n_G = 3, k_G = 2, n_H = 3, k_H = 2 \)

**Theorem (Gaar, Krenn, Margulies, W. 2019)**

- For \( k_G = n_G - 1, n_H = 3 \) and \( k_H = 2 \)

Vizing’s conjecture is true as the polynomials

\[
sg = \frac{1}{3} \left( \sum_{h \in V(H)} x_{gh} - 2 \sum_{\{h, h'\} \subseteq V(H)} x_{gh}x_{gh'} \right) \quad \text{for } g \in V(G)
\]

\[
s_0 = \alpha + \beta \left( \sum_{g \in V(G)} \sum_{h \in V(H)} x_{gh} \right) + \gamma \left( \sum_{g \in V(G)} \sum_{\{h, h'\} \subseteq V(H)} x_{gh}x_{gh'} \right)
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where \( \alpha = \sqrt{2}(n_G - 1), \beta = -\frac{2}{3}\sqrt{2} \) and \( \gamma = \frac{1}{3}\sqrt{2} \)

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Further certificates I

Theorem (Gaar, Krenn, Margulies, W. 2019)

- For $k_G = n_G \geq 1$ and $k_H = n_H - 1 \geq 1$

Vizing’s conjecture is true as the polynomials

$$s_g = \left( \sum_{h \in V(H)} x_{gh} \right) - k_H \quad \text{for } g \in V(G)$$

are a 1-sos certificate of $f^*$. 
Further certificates II

Theorem (Gaar, Krenn, Margulies, W. 2019)

- For \( k_G = n_G \geq 1 \) and \( k_H = n_H - 2 \geq 1 \)
  
  Vizing’s conjecture is true as the polynomials

\[
s_g = \alpha + \beta \left( \sum_{h \in V(H)} x_{gh} \right) + \gamma \left( \sum_{\{h, h'\} \subseteq V(H)} x_{gh}x_{gh'} \right)
\]

for \( g \in V(G) \)

- with

\[
\alpha = (n_H - 2)(n_H + \frac{1}{2}(n_H - 1)\sqrt{2}) \\
\beta = -((2n_H - 3) + (n_H - 2)\sqrt{2}) \\
\gamma = 2 + \sqrt{2}
\]

are a 2-sos certificate of \( f^* \).
Further certificates III

Theorem (Gaar, Krenn, Margulies, W. 2019)

▶ For \( k_G = n_G \geq 1 \) and \( k_H = n_H - 3 \geq 1 \)

Vizing’s conjecture is true as the polynomials

\[
s_g = \sum_{i=0}^{3} \alpha_i \sum_{S \subseteq V(H)} \prod_{h \in S} x_{gh} \quad \text{for} \quad g \in V(G)
\]

▶ with

\[
\alpha_0 = -\frac{1}{6} n^3_H \left( \sqrt{3} + 3 \sqrt{2} + 3 \right) + \frac{1}{2} n^2_H \left( 2 \sqrt{3} + 5 \sqrt{2} + 4 \right) - \frac{1}{2} n_H \left( \frac{11}{3} \sqrt{3} + 6 \sqrt{2} + 3 \right) + \sqrt{3},
\]

\[
\alpha_1 = +\frac{1}{2} n^2_H \left( \sqrt{3} + 3 \sqrt{2} + 3 \right) - \frac{1}{2} n_H \left( 5 \sqrt{3} + 13 \sqrt{2} + 11 \right) + 3 \left( \sqrt{3} + 2 \sqrt{2} \right) + 4,
\]

\[
\alpha_2 = -n_H \left( \sqrt{3} + 3 \sqrt{2} + 3 \right) + 3 \sqrt{3} + 8 \sqrt{2} + 7,
\]

\[
\alpha_3 = \sqrt{3} + 3 \sqrt{2} + 3
\]

are a 3-sos certificate of \( f^* \).
Further certificates IV

Theorem (Gaar, Krenn, Margulies, W. 2019)

For $k_G = n_G \geq 1$ and $k_H = n_H - 4 \geq 1$ Vizing’s conjecture is true as

$$s_g = \sum_{i=0}^{4} \alpha_i \sum_{S \subseteq V(H)} \prod_{h \in S} x_{gh} \quad \text{for } g \in V(G) \text{ with}$$

$$\alpha_0 = \frac{1}{12} n_H^4 \left( 2 \sqrt{3} + 3 \sqrt{2} + 1 \right) - \frac{1}{6} n_H^3 \left( 9 \sqrt{3} + 12 \sqrt{2} + 2 \right)$$
$$+ \frac{1}{12} n_H^2 \left( 52 \sqrt{3} + 57 \sqrt{2} - 7 \right) - \frac{1}{6} n_H \left( 24 \sqrt{3} + 18 \sqrt{2} - 17 \right) - 2,$$

$$\alpha_1 = -\frac{1}{3} n_H^3 \left( 2 \sqrt{3} + 3 \sqrt{2} + 1 \right) + \frac{1}{2} n_H^2 \left( 11 \sqrt{3} + 15 \sqrt{2} + 3 \right)$$
$$- \frac{1}{6} n_H \left( 83 \sqrt{3} + 99 \sqrt{2} + 1 \right) + 10 \sqrt{3} + 10 \sqrt{2} - 3,$$

$$\alpha_2 = n_H^2 \left( 2 \sqrt{3} + 3 \sqrt{2} + 1 \right) - n_H \left( 13 \sqrt{3} + 18 \sqrt{2} + 4 \right) + 5 \left( 4 \sqrt{3} + 5 \sqrt{2} \right) + 2,$$

$$\alpha_3 = -2 n_H \left( 2 \sqrt{3} + 3 \sqrt{2} + 1 \right) + 15 \sqrt{3} + 21 \sqrt{2} + 5,$$

$$\alpha_4 = 4 \sqrt{3} + 6 \sqrt{2} + 2$$

are a 4-sos certificate of $f^*$. 
<table>
<thead>
<tr>
<th>1 - Graph Theory</th>
<th>2 - Split</th>
<th>3 - Algebraic Model</th>
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</thead>
<tbody>
<tr>
<td>$\gamma(G) \gamma(H) \leq \gamma(G \square H)$</td>
<td>$n_G, k_G, n_H, k_H$</td>
<td>$f^*(z) \geq 0 \ \forall z \in \mathcal{V}(I_{viz})$</td>
</tr>
<tr>
<td>4 - Sum of Squares</td>
<td>5 - Semidefinite Programming</td>
<td>6 - SDP Solver</td>
</tr>
<tr>
<td>$f^* \equiv \sum_{i=1}^{t} s_i^2 \mod I_{viz}$</td>
<td>$\exists X : X \succeq 0, \quad f^* \equiv v^T X v \mod I_{viz}$</td>
<td>numeric certificate</td>
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<td>7 - Guess</td>
<td>8 - SAGE</td>
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<td>exact certificate</td>
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</table>
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